Introduction to Probability

BUS 735: Business Decision Methods and Research

1 Goals and Agenda

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Learning Objective	**Active Learning Activity**
Learn basics of probability.	Lecture / Practice Problems
Learn what are probability distributions	Lecture / Practice Problems.
Learn specific probability distributions: Binomial Distribution, Normal Distribution	Lecture / Practice problems.
Learn how to combine random variables.	Lecture / Practice Problems.
Practice what we have learned.	Group Exercise.
More practice.	Read Chapter 11, Homework exercises.
Assess what we have learned	Quiz

2 Basic Probability

2.1 Probability of Events

Basic Probability

- **Probability:** numeric value between 0 and 1 (or 0% and 100%) representing the chance, likelihood, or possibility some event will occur.
- **Event:** some possible (or even impossible) outcome occurring.
  - Denote events with capital English letters.
- Example:
  - A: A newborn baby will be female.
  - $P(A) = 0.5$ means there is a 50% chance that a newborn baby is female.
- Computing probability:

$$P(A) = \frac{n(A)}{T}$$

- $n(A)$ = number of ways event A can occur.
- T = total number of possible outcomes.

Contingency Table

	Purchased DVR		
TV Purchased	Yes	No	Total
HDTV	38	42	80
Regular TV	70	150	220
Total	108	192	300

- Define Event A: Purchased an HDTV.
- What is $P(A)$?

Joint Events

- **Joint Event:** is an event that is composed of two or more events.
- Define event C as any event in either A *or* B.
  - Notation for event C: $C = A \cup B$
  - Notation for probability of event C: $P(C) = P(A \cup B)$.
- Define event C as any event in A *and* B.
  - Notation for event C: $C = A \cap B$
  - Notation for probability of event C: $P(C) = P(A \cap B)$.

Complements of Events

- The **complement** of an event, A, is the outcome of anything *besides* A occurring.
- Notation: $A' = A^c$ = complement of event A.
- $P(A') = 1 - P(A)$.
- Example: what is the complement of Event A: a newborn baby is a female.

Mutually Exclusive Events

- Two events, A and B, are **mutually exclusive** if it is impossible for both A and B to occur at the same time.
- Are the following mutually exclusive?
  - Event A: A person is currently 8 years old. Event B: A person voted for Obama in the last presidential election.

- Event A: A person plays football in high school. Event B: A person plays basketball in high school.
- Event A, Event A'.
- Event A: We will go out to eat tonight. Event B: We are going out to eat tomorrow night.

Contingency Table

	Purchased DVR		
TV Purchased	Yes	No	Total
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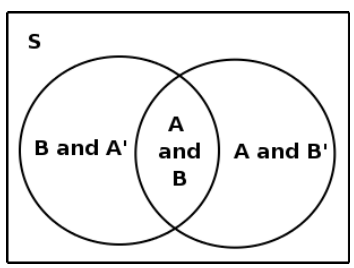
Total	108	192	300

- Define Event A: Purchased an HDTV.
- Define Event B: Purchased a DVR.
- Define Event $C = A \cap B$.
- What is $P(C)$?
- Define Event $D = A \cup B$.
- What is $P(D)$?

2.2 Venn Diagram

Venn Diagram



- **Venn diagram:** visualization of all possible events. The areas in the diagram represent the probabilities of those events.
- S: Event that encompasses all possible outcomes.
- Entire area of the left hand circle is $P(B)$.
- Entire area of the right hand circle is event $P(A)$.
- The area that is in both of the circles is $P(A \cap B)$.

Venn Diagram

- From the Venn Diagram we can see that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Use this equation to find the probability of owning an HDTV (Event A) or owning a DVR (Event B).

	Purchased DVR		
TV Purchased	Yes	No	Total
HDTV	38	42	80
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2.3 Conditional Probability

Conditional probability

- **Conditional probability**: the probability of an event, A, with the additional information that some other event B has already occurred.
- Example:
  - What is the probability of being female?
    Define event A: a person is female.
    $P(A) = 0.5$
  - What is the probability of being female, given you are a nurse?
    Define event B: a person in a nurse.
    $P(A|B) = 0.8$ (I just made that up)

Independence

- Two events A and B are **independent** if knowledge that A happened does not affect the probability that B occurs, or if knowledge that B happened does not effect the probability that A occurs.
- In the example above, is being female and being a nurse independent?
- More examples:
  - Is the event that someone smokes and the event someone has lung cancer independent?
  - Suppose a coin is flipped twice. Is the event the first flip is heads and the event the second flip is heads independent?
- If A and B are independent, then
  $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

Bayes Theorem

- This is the coolest thing you'll ever learn a math class:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Why is the cool? Because this proves that:

$$P(A|B) \neq P(B|A)$$

Bayes Theorem

- Not so cool example: Suppose $P(A) = 0.4$, $P(B) = 0.8$, and $P(A \cap B) = 0.2$. What is $P(A|B)$?

$$P(A|B) = \frac{0.2}{0.8} = 0.25$$

- Are events A and B independent?

Blood test accuracy

- Suppose a fatal disease breaks out, and a blood test is used to detect the disease.
- The blood test claims to accurately identify the disease 99% of the time.
- Let A be the event you have a disease.
- Let B be the event the blood test came out positive.

$$P(B|A) = 0.99$$

- Suppose you take the blood test and it is positive. What is the probability you have the disease?

$$P(A|B) = \ ?$$

Blood test accuracy

- Suppose 0.2% of people have the disease, and 0.198% have the disease and tested positive.

$$P(A) = 0.002$$

$$P(A \cap B) = 0.00198$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.00198}{0.002} = 0.99$$

Blood test accuracy

- Suppose 5% of people test positive for the disease.
- What is the probability you have the disease given you tested positive?

$$P(A|B) = \frac{P(A \cap B)}{P(A)} = \frac{0.00198}{0.05} = 0.0396$$

- Even though you tested positive, *you still most likely do not have the disease.*
- And the test had the claim of being 99% accurate.
- So what is the real accuracy of things like blood tests, pregnancy tests, and lie detector tests?

3 Probability Distributions

3.1 Random Variables

Definitions

- **Random variable**: a variable that has a single numerical value determined by chance.
- Data is a bunch of realizations of a random variable.
- **Discrete random variable**: an RV that can take on "countable" values.
- **Continuous random variable**: a RV that can take on infinitely many values on a continuous scale.

Probability distribution

- A **probability distribution** is a graph, table, or formula that gives the probability for each value of a random variable.
- The sum of the probabilities for all possible values an RV can take must equal 1 (or 100%).

$$\sum P(x_i) = 1$$

- Each individual probability must be between zero and one.

$$0 \leq P(x_i) \leq 1$$

Mean and variance of a probability distribution

- The mean or **expected value** of a probability distribution is:

$$\mu = \sum x_i P(x_i)$$

- The variance of a probability distribution is given by:

$$\sigma^2 = \sum \left[(x_i - \mu)^2 P(x_i)\right]$$

$$\sigma^2 = \sum \left(x_i^2 P(x_i)\right) - \mu^2$$

- Try calculating the mean, variance, and standard deviation for the previous example.

3.2 Binomial distribution

Binomial distributions

- A **Bernoulli trial** results in a random variable that can only result in success ($x = 1$) or failure ($x = 0$).
  - Example: an outcome of heads for a single coin flip is a Bernoulli trial.
- A **binomial distribution** is the probability distribution for the number of successes in a fixed number of trials.
- Requirements for a binomial distribution:
  - The experiment must have a fixed number of Bernoulli trials.
  - The trials must be independent.
  - The probability of success must be the same for each trial.
- Example: What are the possible outcomes for the number of successes in 3 coin flips?

Binomial probability distribution

- The binomial probability distribution is given by,

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

  - n: number of trials.
  - $P(x)$: the probability of x number of successes.
  - p is the probability of success for a single Bernoulli trial.

- Calculate the probability distribution of the 3 coin flip experiment.
- Verify $\sum P(x_i) = 1$.
- Calculate the expected value for the number of heads.
- Calculate the variance and standard deviation.

Binomial Distribution

- Recall the mean of a probability distribution:

$$\mu = \sum x_i P(x_i)$$

- For the binomial distribution, this gets more simple:

$$\mu = np$$

- Recall the variance of a probability distribution:

$$\sigma^2 = \sum \left[(x_i - \mu)^2 P(x_i)\right]$$

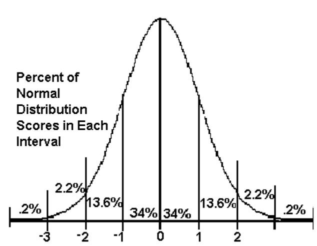
- It gets simpler:

$$\sigma^2 = np(1-p)$$

3.3 Normal Distribution

Normal Distribution

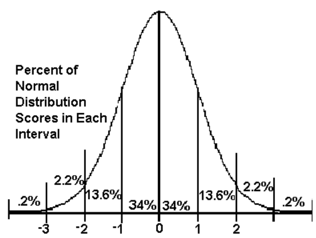
- A very specific symmetric "bell shaped" curve that predicts precise probabilities for ranges of values.
- Probabilities depend on how far an observation is away from the mean.



- Horizontal Axis: number of standard deviations away from the mean.
- Area under the curve represents probability.
- Formula: $f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Normal Distribution

German mathemetician and scientist Carl Friedrich Gauss (1777-1855) derived the normal distribution.



$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Deutsche Mark

Computing Normal Probabilities Using Excel

- `=normsdist('val')` returns the $P(z <' val')$.
- Examples: Suppose Shep's Shoe Shop November sales revenue is normally distributed and has a mean of \$3,500 with a standard deviation of \$800.
  - Suppose Shep's monthly fixed costs are \$2,000. What is the probability November sales fail to cover fixed costs?
  - Shep only has enough inventory to sell \$4,500 worth of shoes. What is the probability his sales will exceed his inventory?

Normal Approximation to a Binomial

- Motivation: Questions like,
- Suppose a study concluded that Burger King fills out 10% of its orders inaccurately. If a particular franchise makes 30,000 orders a month, what is the probability it will make more than 3,100 errors in orders?
- Compute P(x=3100), P(x=3101), P(x=3102), P(x=3103), ... , P(x=30,000).

- That's a ridiculous amount of computations.
- Can you suppose that the number of errors is *normally distributed* with mean equal to $\mu = np$, and variance $\sigma^2 = np(1-p)$?

Normal Approximation to a Binomial

- Can you suppose that the number of errors is *normally distributed* with mean equal to $\mu = np$, and variance $\sigma^2 = np(1-p)$?
- Well... the number of errors is truly a *binomial distribution*, not a normal distribution.
- And... the number of errors is a *discrete* random variable, but the normal distribution is for continuous random variables.
- But... the normal distribution will be a good approximation if,

$$np > 5 \text{ and } n(1-p) > 5$$

4 Combining Random Variables

4.1 Covariance

Covariance

- To measure the variance of combinations of RVs, need to know the covariance.
- **Covariance**: measure of how two RVs move together.
- Notation:
  - σ_{xy}: population covariance.
  - s_{xy}: sample covariance.
- Interpretations:
  - When covariance is negative, variables move in opposite directions.
  - When covariance is positive, variables move in same direction.

4.2 Adding Random Variables

Combining Random Variables

- Suppose two RVs are measured in the same units, can they be combined (added, subtracted, etc)?
- Who would want to do such a thing?

- Addition rule:
  - If $Z = X + Y$, then $E(Z) = E(X) + E(Y)$.
  - More generally, if $Z = aX + bY$, then $E(Z) = aE(X) + bE(Y)$.

Variance of Combinations

- Suppose $Z = X + Y$:

$$VAR(X + Y) = VAR(X) + VAR(Y) + 2COV(X, Y).$$

- Suppose $Z = aX + bY$:

$$VAR(aX + bY) = a^2VAR(X) + b^2VAR(Y) + 2abCOV(X, Y).$$

4.3 Example: Portfolio Risk

Portfolio Risk

- Don't put all your eggs in one basket.
- Would it make more sense to put your money in two investments that are negatively correlated (meaning they have a negative covariance)?
- Example, suppose the variance for the quarterly return is equal to 15% for Investment X and 10% for Investment Y, and the covariance is equal to -8%. Suppose you invested have your money in each investment.
- Would it make more sense to put your money in two investments that are positively correlated (meaning they have a positive covariance)?
- Example, suppose the variance for the quarterly return is equal to 18% for Investment X and 12% for Investment Y, and the covariance is equal to 6%. Suppose you invested have your money in each investment.

5

5.1 Next time on Business Decision Making and Research...

Next time...

- (Re)read the textbook on this topic (BWT, Chapter 11).
- Homework assignment: End of Chapter 11 problems 7, 9, 11c, 13c, 19, 21, 31, 33, 37.
- Quiz on this topic.
- Next topic: Decision Analysis (BWT, Chapter 12).